Math 3063 Abstract Algebra Project 3 Name:
Prof. Paul Bailey March 5, 2007

Complete these problems and turn in the solution by Monday, March 12, 2007. Attach this page to the front of the solutions. Solutions should be self explanatory and written in complete sentences.

Groups

Definition 1. A *group* $(G, \cdot)$ is a set G together with a binary operation

$$\cdot : G \times G \to G$$

such that

(G1) $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for every $g_1, g_2, g_3 \in G$;

(G2) there exists $1 \in G$ such that $g \cdot 1 = 1 \cdot g = g$ for every $g \in G$;

(G3) for every $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.

Problem 1. Let X be a set and define an binary operation

$$\triangle : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X) \quad \text{by} \quad A \triangle B = (A \cup B) \smallsetminus (A \cap B).$$

Show that $(\mathcal{P}(X), \triangle)$ is a group.

Definition 2. A group G is *abelian* if $g_1 g_2 = g_2 g_1$ for every $g_1, g_2 \in G$.

Problem 2. Let G be a group such that $g^2 = 1$ for every $g \in G$. Show that G is abelian.

Definition 3. Let G be a group and let $H \subset G$. We say that H is a *subgroup* of G, and write $H \leq G$, if

(S0) H is nonempty;

(S1) $h_1, h_2 \in H$ implies $h_1 h_2 \in H$;

(S2) $h \in H$ implies $h^{-1} \in H$.

Problem 3. Let G be a group. The *center* of G is

$$Z(G) = \{h \in G \mid hg = gh \text{ for all } g \in G\}.$$

Show that $Z(G) \leq G$.

Problem 4. Let G be a group and let $g_1, g_2 \in G$. We say that g_1 is *conjugate* to g_2, and write $g_1 \sim g_2$, if there exists $h \in G$ such that $g_1 h = h g_2$.

(a) Show that $\sim$ is an equivalence relation.

(b) List the equivalence classes of $\sim$ on S_4.

Problem 5. Let T denote a regular tetrahedron, inscribed in a sphere. Label the vertices 1, 2, 3, 4. Let G denote the set of rotations of the sphere which permute the vertices; we may view G as a subgroup of S_4. List the elements of G in disjoint cycle notation, and construct a Cayley table for the group G.